Pole (complex analysis)

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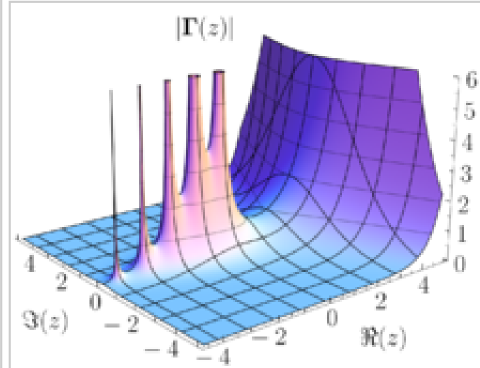
In the mathematical field of complex analysis, a **pole** of a meromorphic function is a certain type of singularity that behaves like the singularity of $\frac{1}{z^n}$ at $z = 0$. For a pole of the function $f(z)$ at point a the function approaches infinity as z approaches a.



The absolute value of the Gamma function. This shows that a function becomes infinite at the poles (left, for negative real values). On the right, the Gamma function does not have poles, it just increases quickly.

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Definition

Formally, suppose U is an open subset of the complex plane **C**, p is an element of U and $f : U \setminus \{p\} \to \mathbf{C}$ is a function which is holomorphic over its domain. If there exists a holomorphic function $g : U \to \mathbf{C}$, such that $g(p)$ is nonzero, and a positive integer n, such that for all z in $U \setminus \{p\}$

$$f(z) = \frac{g(z)}{(z-p)^n}$$

holds, then p is called a **pole of *f***. The smallest such n is called the **order of the pole**. A pole of order 1 is called a **simple pole**.

A few authors allow the order of a pole to be zero, in which case a pole of order zero is either a regular point or a removable singularity. However, it is more usual to require the order of a pole to be positive.

From above several equivalent characterizations can be deduced:

If n is the order of pole p, then necessarily $g(p) \neq 0$ for the function g in the above expression. So we can put

$$f(z) = \frac{1}{h(z)}$$

for some h that is holomorphic in an open neighborhood of p and has a zero of order n at p. So informally one might say that poles occur as reciprocals of zeros of holomorphic functions.

Also, by the holomorphy of g, f can be expressed as:

$$f(z) = \frac{a_{-n}}{(z-p)^n} + \cdots + \frac{a_{-1}}{(z-p)} + \sum_{k \geq 0} a_k (z-p)^k.$$

This is a Laurent series with finite *principal part*. The holomorphic function $\sum_{k \geq 0} a_k (z-p)^k$ (on U) is called the *regular part* of f. So the point p is a pole of order n of f if and only if all the terms in the Laurent series expansion of f around p below degree $-n$ vanish and the term in degree $-n$ is not zero.

Pole at infinity

A complex function can be defined as having a pole at the point at infinity. In this case U has to be a neighborhood of infinity, such as the exterior of any closed ball. To use the previous definition, a meaning for g being holomorphic at ∞ is needed. Alternately, a definition can be given starting from the definition at a finite point by suitably mapping the point at infinity to a finite point. The map $z \mapsto \frac{1}{z}$ does that. Then, by definition, a function f holomorphic in a neighborhood of infinity has a pole at infinity if the function $f(\frac{1}{z})$ (which will be holomorphic in a neighborhood of $z=0$), has a pole at $z=0$, the order of which will be regarded as the order of the pole of f at infinity.

Pole of a function on a complex manifold

In general, having a function $f\colon M \to \mathbb{C}$ that is holomorphic in a neighborhood, U, of the point a, in the complex manifold M, it is said that f has a pole at a of order n if, having a chart $\phi\colon U \to \mathbb{C}$, the function $f \circ \phi^{-1}\colon \mathbb{C} \to \mathbb{C}$ has a pole of order n at $\phi(a)$ (which can be taken as being zero if a convenient choice of the chart is made).] The pole at infinity is the simplest nontrivial example of this definition in which M is taken to be the Riemann sphere and the chart is taken to be $\phi(z) = \frac{1}{z}$.

Examples

- The function

  $$f(z) = \frac{3}{z}$$

  has a pole of order 1 or simple pole at $z = 0$.

- The function

  $$f(z) = \frac{z+2}{(z-5)^2(z+7)^3}$$

  has a pole of order 2 at $z = 5$ and a pole of order 3 at $z = -7$.

- The function

  $$f(z) = \frac{z-4}{e^z - 1}$$

has poles of order 1 at $z = 2\pi n i$ for $n = \ldots, -1, 0, 1, \ldots$. To see that, write e^z in Taylor series around the origin.

- The function

$$f(z) = z$$

has a single pole at infinity of order 1.

Terminology and generalizations

If the first derivative of a function *f* has a simple pole at *a*, then *a* is a branch point of *f*. (The converse need not be true).

A non-removable singularity that is not a pole or a branch point is called an essential singularity.

A complex function which is holomorphic except for some isolated singularities and whose only singularities are poles is called meromorphic.

See also

- Control theory#Stability
- Filter design
- Filter (signal processing)
- Nyquist stability criterion
- Pole–zero plot
- Residue (complex analysis)
- Zero (complex analysis)

External links

- Weisstein, Eric W., "Pole" (http://mathworld.wolfram.com/Pole.html), *MathWorld*.
- Module for Zeros and Poles by John H. Mathews (http://math.fullerton.edu/mathews/c2003/SingularityZeroPoleMod.html)

Categories: Meromorphic functions